

INVARIANCE PRINCIPLE FOR TEMPERED FRACTIONAL TIME SERIES MODELS

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ABSTRACT. Autoregressive tempered fractionally integrated moving average (ARTFIMA) time series is a useful model for velocity data in turbulence flows. In this paper, we obtain an invariance principle for the partial sum of an ARTFIMA process. The limiting process is called tempered Hermite process of order one, THP^1 , which is well-defined for any $H > \frac{1}{2}$. When $\frac{1}{2} < H < 1$, we develop the Wiener integral with respect to THP^1 to provide the sufficient condition for the convergence

$$n^{-H} \sum_{k=0}^{+\infty} f\left(\frac{k}{n}\right) X_k^{\frac{\lambda}{n}} \rightarrow \int_{\mathbb{R}} f(u) Z_{H,\lambda}^1(du)$$

in distribution, as $n \rightarrow \infty$, where X_k is an ARTFIMA time series and $Z_{H,\lambda}^1$ is THP^1 .

1. INTRODUCTION

The motivation of this work comes from the application of stochastic processes in the theory of turbulence. Kolmogorov [18, 11] proposed a model for the energy spectrum of turbulence in the inertial range, predicting that the spectrum $f(k)$ would follow a power law $f(k) \propto k^{-5/3}$ where k is the frequency.

Figure 1 illustrates the complete Kolmogorov spectral model for turbulence, and the power law approximation in the inertial range. Large eddies are produced in the low frequency range. In the inertial range, larger eddies are continuously broken down into smaller eddies, until they eventually dissipate, in the high frequency range.

The autoregressive tempered fractionally integrated moving average (ARTFIMA) time series modifies the coefficient of an autoregressive tempered fractionally integrated moving average (ARFIMA) model by multiplying an exponential tempering factor. The spectral density of the ARTFIMA $(0, \alpha, \lambda, 0)$ is proportional to $|e^{-(\lambda+ik)} - 1|^{-2\alpha} \approx (\lambda^2 + k^2)^{-\alpha}$ when k, λ are sufficiently small. For small values of the tempering parameter λ , the spectral density of an ARTFIMA $(0, \alpha, \lambda, 0)$ time series grows like $k^{-2\alpha}$ as $|k|$ decreases, but remains bounded as $|k| \rightarrow 0$, in agreement with the general theory of turbulence illustrated in Figure 1. We refer the reader to [25] to see the application of the ARTFIMA time series for turbulence in geophysical flows.

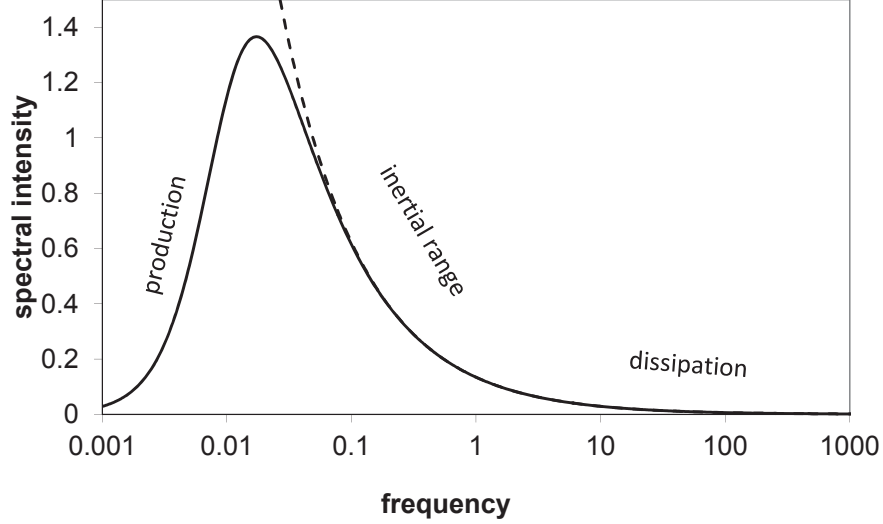


FIGURE 1. Kolmogorov spectral density (solid line) and power law approximation in the inertial range (dotted line), from [25].

As it was mentioned, the ARTFIMA time series can be a useful discrete time stochastic model for turbulence. In this paper, we are interested to answer several questions which are related with the ARTFIMA time series. The first question is:

- 1:** Assume X_k follows an ARTFIMA time series model. When do we have an invariance principle

$$n^{-H} \sum_{k=0}^{[nt]} X_k \Rightarrow Y(t)$$

and what is the limiting process Y ?

We prove that $\{Y(t)\}_{t \geq 0}$ is a Gaussian process which interpolates between fractional Brownian motion (FBM) and the standard Ornstein Uhlenbeck process with the time domain representation

$$Y(t) := Z_{H,\lambda}^1(t) = \int_{\mathbb{R}} \int_0^t \left((s-y)_+^{H-\frac{3}{2}} e^{-\lambda(s-y)_+} \right) ds B(dy),$$

where $(x)_+ = xI(x > 0)$, $B(dy)$ is an independently scattered Gaussian random measure on \mathbb{R} with control measure $\sigma^2 dx$, $H > \frac{1}{2}$ and $\lambda > 0$. The process $Z_{H,\lambda}^1$ is called tempered Hermite process of order one, THP^1 .

We called $Z_{H,\lambda}^1$ as tempered Hermite process of order one since it can be extended to

$$Z_{H,\lambda}^k(t) := \int'_{\mathbb{R}^k} \int_0^t \left(\prod_{i=1}^k (s - y_i)_+^{-(\frac{1}{2} + \frac{1-H}{k})} e^{-\lambda(s-y_i)_+} \right) ds B(dy_1) \dots B(dy_k)$$

for any $k \geq 2$. The prim on the integral sign shows that one does not integrate on diagonals where $y_i = y_j$, $i \neq j$. The Hermite process [34, 8] is a special case of $\{Z_{H,\lambda}^k\}$ with $\lambda = 0$. Unlike the Hermite process, tempered Hermite process of order k is well-defined also for any $H > \frac{1}{2}$ because the exponential tempering keeps the integrand in $L^2(\mathbb{R}^k)$.

The second question is naturally the extension of the first question:

2: Suppose f is a deterministic function. What is the sufficient condition for the convergence

$$n^{-H} \sum_{k=0}^{+\infty} f\left(\frac{k}{n}\right) X_k \rightarrow \int_{\mathbb{R}} f(u) Z_{H,\lambda}^1(du)$$

in distribution, as $n \rightarrow \infty$?

In order to provide the sufficient condition for the convergence (in distribution) in the second question, we first require to develop the Wiener integral $\int_{\mathbb{R}} f(u) Z_{H,\lambda}^1(du)$, where f is a deterministic functions in an appropriate space and $\frac{1}{2} < H < 1$. Our approach to develop the Wiener integral with respect to $Z_{H,\lambda}^1$ is based on tempered fractional calculus. In fact, we show that a representation of $Z_{H,\lambda}^1$ based on fractional calculus. That is,

$$Z_{H,\lambda}^1(t) = \Gamma(H - \frac{1}{2}) \int_{-\infty}^{+\infty} \left(\mathbb{I}_{-}^{H-\frac{1}{2},\lambda} \mathbf{1}_{[0,t]} \right)(x) B(dx),$$

where $H > \frac{1}{2}$ and $\left(\mathbb{I}_{-}^{\alpha,\lambda} f \right)(x)$ is tempered fractional integral of order α of a function f . We refer the reader to [22] for more details on tempered fractional integrals and derivatives. This representation enables us to characterize the classes of deterministic functions f for which the Wiener integral $\int_{\mathbb{R}} f Z_{H,\lambda}^1(du)$ is well defined. On the other hand, we need to study the asymptotic behavior of the spectral density of the ARTFIMA $(0, \alpha, \lambda, 0)$ for low frequency. Therefor, we can prove the convergence in distribution that we had in the second question.

The paper is organized as follows. In Section 2, we define tempered Hermite process of order one, $\{Z_{H,\lambda}^1\}$, using a time domain representation, and we develop the spectral domain representation of $\{Z_{H,\lambda}^1\}$. In Section 3, we answer the second question by investigating the Wiener integral with respect to tempered Hermite process of order one. In Section 4, we recall the definition of the autoregressive tempered fractionally integrated moving average (ARTFIMA) time series and some of its basic properties such as the covariance function and spectral density. The answer of the first and third

question are given in Section 5. Some definitions and lemmas which are related to fractional calculus are contained in Appendix.

2. TIME AND SPECTRAL DOMAIN REPRESENTATIONS

In this section, we define tempered Hermite processes of order one, THP^1 . We start with the time domain representation.

Let $\{B(t)\}_{t \in \mathbb{R}}$ be a real-valued Brownian motion on the real line, a process with stationary independent increments such that $B(t)$ has a Gaussian distribution with mean zero and variance $|t|$ for all $t \in \mathbb{R}$. Define an independently scattered Gaussian random measure $B(dx)$ with control measure $m(dx) = dx$ by setting $B[a, b] = B(b) - B(a)$ for any real numbers $a < b$, and then extending to all Borel sets. Since Brownian motion sample paths are almost surely of unbounded variation, the measure $B(dx)$ is not almost surely σ -additive, but it is a σ -additive measure in the sense of mean square convergence. Then the stochastic integrals $I(f) := \int f(x)B(dx)$ are defined for all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $\int f(x)^2 dx < \infty$, as Gaussian random variables with mean zero and covariance $\mathbb{E}[I(f)I(g)] = \int f(x)g(x)dx$. See for example [33, Chapter 3] or [23, Section 7.6].

Definition 2.1. *Given an independently scattered Gaussian random measure $B(dx)$ on \mathbb{R} with control measure $\sigma^2 dx$, $H > \frac{1}{2}$, $\lambda > 0$, the stochastic integral*

$$(1) \quad Z_{H,\lambda}^1(t) := \int_{\mathbb{R}} \int_0^t \left((s-y)_+^{H-\frac{3}{2}} e^{-\lambda(s-y)_+} \right) ds B(dy)$$

where $(x)_+ = xI(x > 0)$, will be called a tempered Hermit process of order one (THP^1).

The next lemma shows that $Z_{H,\lambda}^1(t)$ is well-defined for any $t > 0$.

Lemma 2.2. *The function*

$$(2) \quad g_{H,\lambda,t}(y) := \int_0^t (s-y)_+^{H-\frac{3}{2}} e^{-\lambda(s-y)_+} ds$$

is square integrable over the entire real line for any $H > \frac{1}{2}$ and $\lambda > 0$.

Proof. The proof is similar to [2, Theorem 3.5]. To show that $g_{H,\lambda,t}(y)$ is square integrable over the entire real line, we write

$$\begin{aligned}
\int_{\mathbb{R}} g_{H,\lambda,t}(y)^2 dy &= \int_{\mathbb{R}} \int_0^t \int_0^t (s_1 - y)_+^{H-\frac{3}{2}} e^{-\lambda(s_1-y)_+} (s_2 - y)_+^{H-\frac{3}{2}} e^{-\lambda(s_2-y)_+} ds_1 ds_2 dy \\
&= 2 \int_0^t ds_1 \int_{s_1}^t ds_2 \int_{\mathbb{R}} (s_1 - y)_+^{H-\frac{3}{2}} e^{-\lambda(s_1-y)_+} (s_2 - y)_+^{H-\frac{3}{2}} e^{-\lambda(s_2-y)_+} dy \\
&= 2 \int_0^t ds \int_0^{t-s} du \int_{\mathbb{R}} (w)_+^{H-\frac{3}{2}} e^{-\lambda(w)_+} (w+u)_+^{H-\frac{3}{2}} e^{-\lambda(w+u)_+} dw \\
&\quad (s = s_1, u = s_2 - s_1, w = s_1 - y) \\
&= 2 \int_0^t ds \int_0^{t-s} u^{2H-2} e^{-\lambda u} du \int_0^{+\infty} x^{H-\frac{3}{2}} (1+x)^{H-\frac{3}{2}} e^{-2\lambda u x} dx \\
&= \frac{2\Gamma(H-\frac{1}{2})}{\sqrt{\pi}(2\lambda)^{H-1}} \int_0^t ds \int_0^{t-s} u^{H-1} K_{1-H}(\lambda u) du,
\end{aligned}$$

where we applied a standard integral formula [12, Page 344]

$$(3) \quad \int_0^\infty x^{\nu-1} (x+\beta)^{\nu-1} e^{-\mu x} dx = \frac{1}{\sqrt{\pi}} \left(\frac{\beta}{\mu}\right)^{\nu-\frac{1}{2}} e^{\frac{\beta\mu}{2}} \Gamma(\nu) K_{\frac{1}{2}-\nu}\left(\frac{\beta\mu}{2}\right),$$

for $|\arg \beta| < \pi$, $\operatorname{Re} \mu > 0$, $\operatorname{Re} \nu > 0$. Here $K_\nu(x)$ is modified Bessel function of the second kind (see Appendix for more details about $K_\nu(x)$). Next, we need to show that the last integrals is finite for any $H > \frac{1}{2}$. First, assume $\frac{1}{2} < H < 1$. In that case, $K_{1-H}(\lambda u) \sim u^{H-1}$ as $u \rightarrow 0$ ([1, Chapter 9]), and hence the integrand $u^{H-1} K_{1-H}(\lambda u) \sim u^{2H-2}$, as $u \rightarrow 0$, which is integrable provided that $H > \frac{1}{2}$. Now, let $H > 1$. In the later case, $K_{1-H}(\lambda u) \sim u^{1-H}$ as $u \rightarrow 0$ and therefore the integrands $u^{H-1} K_{1-H}(\lambda u) \sim C$, C is a constant, which is integrable and this completes the proof. \square

Remark 2.3. When $\lambda = 0$, the right-hand side of (1) is a fractional Brownian motion (FBM), a self-similar Gaussian stochastic process with Hurst scaling index H (e.g., see [9]). When $\lambda = 0$ and $H > 1$, the right-hand side of (1) does not exist, since the integrand is not in $L^2(\mathbb{R})$. However, THP^1 with $\lambda > 0$ and $H > 1$ is well-defined, because the exponential tempering keeps the integrand in $L^2(\mathbb{R})$.

We now compute the covariance function $R(t, s) = \mathbb{E}[Z_{H,\lambda}^1(t) Z_{H,\lambda}^1(s)]$ of THP^1 .

Proposition 2.4. *The process $Z_{H,\lambda}^1$ given by (1) has the covariance function*

$$(4) \quad R(t, s) = \frac{2\Gamma(H-\frac{1}{2})}{\sqrt{\pi}(2\lambda)^{H-1}} \int_0^t \int_0^s |u-v|^{H-1} K_{1-H}(\lambda|u-v|) dv du.$$

Proof. The proof is similar to that of Lemma 2.2. By applying Fubini and the Itô isometry, we have

$$\begin{aligned}
R(t, s) &= 2 \int_{\mathbb{R}} \int_0^t \int_0^s (u-y)_+^{H-\frac{3}{2}} (v-y)_+^{H-\frac{3}{2}} e^{-\lambda(u-y)_+} e^{-\lambda(v-y)_+} dv \, du \, dy \\
&= 2 \int_0^t \int_0^s \int_{-\infty}^v (u-y)^{H-\frac{3}{2}} (v-y)^{H-\frac{3}{2}} e^{\lambda(u-y)} e^{-\lambda(v-y)} dy \, dv \, du \\
&\quad (\text{assume } v < u) \\
&= 2 \int_0^t \int_0^s (u-v)^{2H-2} e^{-\lambda(u-v)} \int_0^{+\infty} x^{H-\frac{3}{2}} (1+x)^{H-\frac{3}{2}} e^{-2\lambda(u-v)x} dx \, dv \, du \\
&\quad (u-y = x(u-v) + u-v) \\
&= \frac{2\Gamma(H-\frac{1}{2})}{\sqrt{\pi}(2\lambda)^{H-1}} \int_0^t \int_0^s (u-v)^{H-1} K_{1-H}(\lambda(u-v)) dv \, du,
\end{aligned}$$

where we applied (3) to get the last integral. Hence

$$R(t, s) = \frac{2\Gamma(H-\frac{1}{2})}{\sqrt{\pi}(2\lambda)^{H-1}} \int_0^t \int_0^s |u-v|^{H-1} K_{1-H}(\lambda|u-v|) dv \, du.$$

By using the same argument in Lemma 2.2, one can show that the covariance is finite for any $H > \frac{1}{2}$. \square

The next results shows that THP^1 has a nice scaling property, involving both the time scale and the tempering. Here the symbol \triangleq indicates the equivalence of finite dimensional distributions.

Proposition 2.5. *The process $Z_{H,\lambda}^1$ given by (2.1) has stationary increments, such that*

$$(5) \quad \{Z_{H,\lambda}^1(ct)\}_{t \in \mathbb{R}} \triangleq \{c^H Z_{H,c\lambda}^1(t)\}_{t \in \mathbb{R}}$$

for any scale factor $c > 0$.

Proof. Since $B(dy)$ has control measure $m(dy) = \sigma^2 dy$, the random measure $B(c \, dy)$ has control measure $c^{1/2} \sigma^2 dy$. Given t_j , $j = 1, \dots, n$, a change of variables $s = cs'$

and $y = cy'$ then yields

$$\begin{aligned}
& (Z_{H,\lambda}^1(ct_j) : j = 1, \dots, n) \\
&= \left(\int_{\mathbb{R}} \int_0^{ct_j} \left((s-y)_+^{H-\frac{3}{2}} e^{-\lambda(s-y)_+} \right) ds B(dy) \right) \\
&= \left(\int_{\mathbb{R}} \int_0^{t_j} \left((cs' - cy)_+^{H-\frac{3}{2}} e^{-\lambda(cs' - cy)_+} \right) c ds' B(dy) \right) \\
&\triangleq c^H \left(\int_{\mathbb{R}} \int_0^{t_j} \left((s' - y')_+^{H-\frac{3}{2}} e^{-\lambda c(s' - y')_+} \right) ds' B(c dy') \right) \\
&= (c^H Z_{H,c\lambda}^1(t_j) : j = 1, \dots, n)
\end{aligned}$$

so that (5) holds. Suppose now $s_j < t_j$, and change variables $x = x' + s$, $y = s + y'$ to get

$$\begin{aligned}
& (Z_{H,\lambda}^1(t_j) - Z_{H,\lambda}^1(s_j) : j = 1, \dots, n) \\
&= \left(\int_{\mathbb{R}} \int_{s_j}^{t_j} \left((x-y)_+^{H-\frac{3}{2}} e^{-\lambda(x-y)_+} \right) dx B(dy) \right) \\
&= \left(\int_{\mathbb{R}} \int_0^{t_j - s_j} \left((x' + s - y)_+^{H-\frac{3}{2}} e^{-\lambda(x' + s - y)_+} \right) dx' B(dy) \right) \\
&\triangleq \left(\int_{\mathbb{R}} \int_0^{t_j - s_j} \left((x' - y')_+^{H-\frac{3}{2}} e^{-\lambda(x' - y')_+} \right) dx' B(dy') \right) \\
&= (Z_{H,\lambda}^1(t_j - s_j) : j = 1, \dots, n)
\end{aligned}$$

which shows that THP^1 has stationary increments. \square

We next give another representation of THP^1 which is called the spectral domain representation. Let \hat{B}_1 and \hat{B}_2 be independent Gaussian random measures with $\hat{B}_1(A) = \hat{B}_1(-A)$, $\hat{B}_2(A) = -\hat{B}_2(-A)$ and $\mathbb{E}[(\hat{B}_i(A))^2] = m(A)/2$, where $m(dx) = \sigma^2 dx$, and define the complex-valued Gaussian random measure $\hat{B} = \hat{B}_1 + i\hat{B}_2$. If $f(x)$ is a complex-valued function of x real such that its Fourier transform $\hat{f}(\omega) := (2\pi)^{-1/2} \int e^{i\omega x} f(x) dx$ exists and $\int |\hat{f}(\omega)|^2 d\omega < \infty$, we define the stochastic integral $\hat{I}(\hat{f}) = \int \hat{f}(\omega) \hat{B}(d\omega) := \int \hat{f}_1(\omega) \hat{B}_1(d\omega) - \int \hat{f}_2(\omega) \hat{B}_2(d\omega)$, where $\hat{f} = \hat{f}_1 + i\hat{f}_2$ is separated into real and imaginary parts. Then $\hat{I}(\hat{f})$ is a Gaussian random variable with mean zero, such that $\mathbb{E}[\hat{I}(\hat{f})\hat{I}(\hat{g})] = \int \hat{f}(\omega) \overline{\hat{g}(\omega)} d\omega$ for all such functions, and the Parseval identity $\int f(x)g(x) dx = \int \hat{f}(\omega) \overline{\hat{g}(\omega)} d\omega$ implies that $(\int f(x)B(dx), \int g(x)B(dx)) \triangleq (\int \hat{f}(\omega)\hat{B}(d\omega), \int \hat{g}(\omega)\hat{B}(d\omega))$, see Proposition 7.2.7 in [33].

Proposition 2.6. *The process $Z_{H,\lambda}^1$ given by (2.1) has the spectral domain representation*

$$(6) \quad Z_{H,\lambda}^1(t) \triangleq \frac{1}{C(H)} \int_{\mathbb{R}} \frac{e^{i\omega t} - 1}{i\omega} (\lambda + i\omega)^{\frac{1}{2}-H} \widehat{B}(d\omega),$$

where $C(H) = \frac{\sqrt{2\pi}}{\Gamma(H-\frac{1}{2})}$.

Proof. To show that the stochastic integral (6) exists, note that $\left| \frac{e^{i\omega t} - 1}{i\omega} (\lambda + i\omega)^{\frac{1}{2}-H} \right|^2$ is bounded for $\omega \rightarrow 0$ and behaves like $|\omega|^{-1-2H}$, as $\omega \rightarrow \infty$, which is integrable provided that $H > 0$. Observe that the function $g_{H,\lambda,t}$, given by (2), has the Fourier transform

$$\begin{aligned} \widehat{g_{H,\lambda,t}}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\omega y} \int_0^t (s-y)_+^{H-\frac{3}{2}} e^{-\lambda(s-y)_+} ds dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\omega y} \int_{\mathbb{R}} (s-y)^{H-\frac{3}{2}} e^{-\lambda(s-y)} \mathbf{1}\{0 < s < t\} \mathbf{1}\{s-y > 0\} ds dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i(s-u)\omega} \int_{\mathbb{R}} u^{H-\frac{3}{2}} e^{-\lambda u} \mathbf{1}\{0 < s < t\} \mathbf{1}\{u > 0\} ds du \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{is\omega} \mathbf{1}\{0 < s < t\} \int_{\mathbb{R}} u^{H-\frac{3}{2}} e^{-(\lambda+i\omega)u} \mathbf{1}\{u > 0\} du ds \\ &= \frac{\Gamma(H-\frac{1}{2})}{\sqrt{2\pi}} \frac{e^{i\omega t} - 1}{i\omega} (\lambda + i\omega)^{\frac{1}{2}-H} \end{aligned}$$

provided that $H > \frac{1}{2}$ and then by applying (1)

$$\begin{aligned} Z_{H,\lambda}^1(t) &= \int_{-\infty}^{+\infty} g_{H,\lambda,t}(x) B(dx) \\ &\triangleq \int_{-\infty}^{+\infty} \widehat{g_{H,\lambda,t}}(\omega) \widehat{B}(d\omega) = \frac{1}{C(H)} \int_{-\infty}^{+\infty} \frac{e^{i\omega t} - 1}{i\omega} (\lambda + i\omega)^{\frac{1}{2}-H} \widehat{B}(d\omega) \end{aligned}$$

which is equivalent to (6). \square

Remark 2.7. We called the process $Z_{H,\lambda}^1$ as the tempered Hermite process of order one, since it is a special case of the following stochastic process which is called tempered Hermite process of order k :

$$(7) \quad Z_{H,\lambda}^k(t) := \int_{\mathbb{R}^k} \int_0^t \left(\prod_{i=1}^k (s-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})} e^{-\lambda(s-y_i)_+} \right) ds B(dy_1) \dots B(dy_k)$$

for any $k \geq 1$ and $H > \frac{1}{2}$. It is easy to check that $Z_{H,\lambda}^k$ has stationary increment with the scaling property given by (5). Moreover, one can verify that $Z_{H,\lambda}^k$ has the

spectral domain representation

$$(8) \quad Z_{H,\lambda}^k(t) = c(H, k) \int_{\mathbb{R}^k}'' \frac{e^{it(\omega_1 + \dots + \omega_k)} - 1}{i(\omega_1 + \dots + \omega_k)} \prod_{j=1}^k (\lambda + i\omega_j)^{-\left(\frac{1}{2} - \frac{1-H}{k}\right)} \widehat{B}(d\omega_1) \dots \widehat{B}(d\omega_k),$$

where $c(H, k) = \left(\frac{\Gamma(\frac{1}{2} - \frac{1-H}{k})}{\sqrt{2\pi}}\right)^k$ is a constant depending on H and k . The double prim on the integral indicates that one does not integrate on diagonals where $\omega_i = \omega_j$, $i \neq j$. In this paper, we just consider tempered Hermite process of order one.

Finally, we close this section with introducing tempered Hermite noise which is the increment of tempered Hermite process of order one. Given a THP^1 , (1), we define tempered Hermite noise (THN)

$$(9) \quad X_n = Z_{H,\lambda}^1(n+1) - Z_{H,\lambda}^1(n) \quad \text{for integers } 0 < n < \infty.$$

It follows easily from (1) that THN has the time domain representation

$$(10) \quad X_n = \int_{\mathbb{R}} \int_n^{n+1} (s-y)_+^{\frac{3}{2}-H} e^{-\lambda(s-y)+} ds B(dy).$$

Using (6), it also follows that THN has the spectral domain representation,

$$(11) \quad X_n = \frac{1}{C(H)} \int_{\mathbb{R}} e^{in\omega} \frac{e^{i\omega} - 1}{i\omega} (\lambda + i\omega)^{\frac{1}{2}-H} \widehat{B}(d\omega).$$

It follows from (11) that THN is a stationary Gaussian time series with mean zero and covariance function

$$(12) \quad r(n) := \mathbb{E}[X_0 X_n] = \frac{\sigma^2}{C(H)^2} \int_{\mathbb{R}} e^{in\omega} \left| \frac{e^{i\omega} - 1}{i\omega} \right|^2 (\lambda^2 + \omega^2)^{\frac{1}{2}-H} (d\omega).$$

Proposition 2.8. *THP (9) has the spectral density*

$$(13) \quad h(\omega) = \frac{1}{C(H)^2} \left| \frac{e^{i\omega} - 1}{i\omega} \right|^2 \sum_{\ell=-\infty}^{+\infty} \sigma^2 [\lambda^2 + (\omega + 2\pi\ell)^2]^{\frac{1}{2}-H}.$$

Proof. Recall that the spectral density

$$(14) \quad h(\omega) = \frac{1}{2\pi} \sum_{j=-\infty}^{+\infty} e^{-i\omega n} r(n) \quad \text{and} \quad r(n) = \int_{-\pi}^{\pi} e^{i\omega n} h(\omega) d\omega.$$

Apply (12) to write

$$(15) \quad \begin{aligned} r(n) &= \frac{\sigma^2}{C(H)^2} \int_{-\infty}^{+\infty} e^{i\omega n} \left| \frac{e^{i\omega} - 1}{i\omega} \right|^2 (\lambda^2 + \omega^2)^{\frac{1}{2}-H} d\omega \\ &= \frac{1}{C(H)^2} \int_{-\pi}^{+\pi} e^{i\omega n} \left| \frac{e^{i\omega} - 1}{i\omega} \right|^2 \sum_{\ell=-\infty}^{+\infty} \sigma^2 [\lambda^2 + (\omega + 2\pi\ell)^2]^{\frac{1}{2}-H} d\omega \end{aligned}$$

and then it follows from (14) that the spectral density of THN is given by (13). \square

Remark 2.9. Extending the definition (9) to all n real positive, we obtain the continuous parameter THN

$$X_t = Z_{H,\lambda}^1(t+1) - Z_{H,\lambda}^1(t).$$

The spectral domain representation of this process is given by (11) with n replaced by t , and the proof of Proposition 2.8 implies that X_t has spectral density

$$(16) \quad h(\omega) = \frac{\sigma^2}{C(H)^2} \left| \frac{e^{i\omega} - 1}{i\omega} \right|^2 [\lambda^2 + \omega^2]^{\frac{1}{2}-H}$$

for all real ω . The fact that $\left| \frac{e^{i\omega} - 1}{i\omega} \right|$ is bounded as $\omega \rightarrow 0$ yields the low frequency approximation

$$(17) \quad h(\omega) \approx \frac{\sigma^2}{C(H)^2} (\lambda^2 + \omega^2)^{\frac{1}{2}-H}.$$

By taking $H = \frac{4}{3}$ in (17), we get $h(\omega) \approx \omega^{-5/3}$ which is the spectral model suggested by Kolmogorov [18, 11] for the energy spectrum of turbulence in the inertial range. The spectral density of THN has some applications in turbulent flows [25].

3. WIENER INTEGRALS WITH RESPECT TO TEMPERED HERMITE PROCESS OF ORDER ONE

In order to get the main results of this paper, Section 5, we need to develop the Wiener integrals with respect to $Z_{H,\lambda}^1$. We consider two cases:

- $\frac{1}{2} < H < 1, \lambda > 0$
- $H > 1, \lambda > 0$

We start with the first case. We first establish a link between $Z_{H,\lambda}^1$ and tempered fractional calculus.

Lemma 3.1. *For a tempered Hermite process of order one given by (1), THP^1 , with $\lambda > 0$, we have:*

$$(18) \quad Z_{H,\lambda}^1(t) = \Gamma(H - \frac{1}{2}) \int_{-\infty}^{+\infty} \left(\mathbb{I}_-^{H-\frac{1}{2},\lambda} \mathbf{1}_{[0,t]} \right)(x) B(dx)$$

where $H > \frac{1}{2}$.

Proof. Write the kernel function from (2) in the form

$$\begin{aligned} g_{H,\lambda,t}(x) &= \int_0^t (s-x)_+^{H-\frac{3}{2}} e^{-\lambda(s-x)_+} ds \\ &= \int_{-\infty}^{+\infty} \mathbf{1}_{[0,t]}(s) (s-x)_+^{H-\frac{3}{2}} e^{-\lambda(s-x)_+} ds \\ &= \Gamma(H - \frac{1}{2}) \left(\mathbb{I}_-^{H-\frac{1}{2},\lambda} \mathbf{1}_{[0,t]} \right)(x) \end{aligned}$$

which gives the desired result. □

Next we discuss a general construction for stochastic integrals with respect to THP^1 . For a standard Brownian motion $\{B(t)\}_{t \in \mathbb{R}}$ on (Ω, \mathcal{F}, P) , the stochastic integral $\mathcal{I}(f) := \int f(x)B(dx)$ is defined for any $f \in L^2(\mathbb{R})$, and the mapping $f \mapsto \mathcal{I}(f)$ defines an isometry from $L^2(\mathbb{R})$ into $L^2(\Omega)$, called the *Itô isometry*:

$$(19) \quad \langle \mathcal{I}(f), \mathcal{I}(g) \rangle_{L^2(\Omega)} = \text{Cov}[\mathcal{I}(f), \mathcal{I}(g)] = \int f(x)g(x) dx = \langle f, g \rangle_{L^2(\mathbb{R})}.$$

Since this isometry maps $L^2(\mathbb{R})$ onto the space $\overline{\text{Sp}}(B) = \{\mathcal{I}(f) : f \in L^2(\mathbb{R})\}$, we say that these two spaces are isometric. For any elementary function (step function)

$$(20) \quad f(u) = \sum_{i=1}^n a_i \mathbf{1}_{[t_i, t_{i+1})}(u),$$

where a_i, t_i are real numbers such that $t_i < t_j$ for $i < j$, it is natural to define the stochastic integral

$$(21) \quad \mathcal{I}^{\alpha, \lambda}(f) = \int_{\mathbb{R}} f(x) Z_{H, \lambda}^1(dx) = \sum_{i=1}^n a_i [Z_{H, \lambda}^1(t_{i+1}) - Z_{H, \lambda}^1(t_i)],$$

and then it follows immediately from (18) that for $f \in \mathcal{E}$, the space of elementary functions, the stochastic integral

$$\mathcal{I}^{\alpha, \lambda}(f) = \int_{\mathbb{R}} f(x) Z_{H, \lambda}^1(dx) = \Gamma(H - \frac{1}{2}) \int_{\mathbb{R}} \left(\mathbb{I}_-^{H - \frac{1}{2}, \lambda} f \right)(x) B(dx)$$

is a Gaussian random variable with mean zero, such that for any $f, g \in \mathcal{E}$ we have

$$(22) \quad \begin{aligned} \langle \mathcal{I}^{\alpha, \lambda}(f), \mathcal{I}^{\alpha, \lambda}(g) \rangle_{L^2(\Omega)} &= \mathbb{E} \left(\int_{\mathbb{R}} f(x) Z_{H, \lambda}^1(dx) \int_{\mathbb{R}} g(x) Z_{H, \lambda}^1(dx) \right) \\ &= \Gamma(H - \frac{1}{2})^2 \int_{\mathbb{R}} \left(\mathbb{I}_-^{H - \frac{1}{2}, \lambda} f \right)(x) \left(\mathbb{I}_-^{H - \frac{1}{2}, \lambda} g \right)(x) dx, \end{aligned}$$

in view of (18) and the Itô isometry (19). The linear space of Gaussian random variables $\{\mathcal{I}^{\alpha, \lambda}(f), f \in \mathcal{E}\}$ is contained in the larger linear space

$$(23) \quad \overline{\text{Sp}}(Z_{H, \lambda}^1) = \{X : \mathcal{I}^{\alpha, \lambda}(f_n) \rightarrow X \text{ in } L^2(\Omega) \text{ for some sequence } (f_n) \text{ in } \mathcal{E}\}.$$

An element $X \in \overline{\text{Sp}}(Z_{H, \lambda}^1)$ is mean zero Gaussian with variance

$$\text{Var}(X) = \lim_{n \rightarrow \infty} \text{Var}[\mathcal{I}^{\alpha, \lambda}(f_n)],$$

and X can be associated with an equivalence class of sequences of elementary functions (f_n) such that $\mathcal{I}^{\alpha, \lambda}(f_n) \rightarrow X$ in $L^2(\mathbb{R})$. If $[f_X]$ denotes this class, then X can be written in an integral form as

$$(24) \quad X = \int_{\mathbb{R}} [f_X] dZ_{H, \lambda}^1$$

and the right hand side of (24) is called the stochastic integral with respect to THP^1 on the real line (see, for example, Huang and Cambanis [14], page 587). In the special case of a Brownian motion $\lambda = 0, H = \frac{1}{2}$, $\mathcal{I}^{\alpha, \lambda}(f_n) \rightarrow X$ along with the Itô isometry (19) implies that (f_n) is a Cauchy sequence, and then since $L^2(\mathbb{R})$ is a (complete) Hilbert space, there exists a unique $f \in L^2(\mathbb{R})$ such that $f_n \rightarrow f$ in $L^2(\mathbb{R})$, and we can write $X = \int_{\mathbb{R}} f(x)B(dx)$. However, if the space of integrands is not complete, then the situation is more complicated. Here we investigate stochastic integrals with respect to THP^1 based on time domain representation. Equation (22) suggests the appropriate space of integrands for THP^1 , in order to obtain a nice isometry that maps into the space $\overline{\text{Sp}}(Z_{H, \lambda}^1)$ of stochastic integrals.

Theorem 3.2. *Given $\frac{1}{2} < H < 1$ and $\lambda > 0$, the class of functions*

$$(25) \quad \mathcal{A}_1 := \left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} \left| \left(\mathbb{I}_-^{H-\frac{1}{2}, \lambda} f \right)(x) \right|^2 dx < \infty \right\},$$

is a linear space with inner product

$$(26) \quad \langle f, g \rangle_{\mathcal{A}_1} := \langle F, G \rangle_{L^2(\mathbb{R})}$$

where

$$(27) \quad \begin{aligned} F(x) &= \Gamma(H - \frac{1}{2}) \left(\mathbb{I}_-^{H-\frac{1}{2}, \lambda} f \right)(x) \\ G(x) &= \Gamma(H - \frac{1}{2}) \left(\mathbb{I}_-^{H-\frac{1}{2}, \lambda} g \right)(x). \end{aligned}$$

The set of elementary functions \mathcal{E} is dense in the space \mathcal{A}_1 . The space \mathcal{A}_1 is not complete.

Proof. The proof is similar to [22, Theorem 3.5]. To show that \mathcal{A}_1 is an inner product space, we will check that $\langle f, f \rangle_{\mathcal{A}_1} = 0$ implies $f = 0$ almost everywhere. If $\langle f, f \rangle_{\mathcal{A}_1} = 0$, then in view of (26) and (27) we have $\langle F, F \rangle_2 = 0$, so $F(x) = \Gamma(H - \frac{1}{2}) \left(\mathbb{I}_-^{H-\frac{1}{2}, \lambda} f \right)(x) = 0$ for almost every $x \in \mathbb{R}$. Then

$$(28) \quad \left(\mathbb{I}_-^{H-\frac{1}{2}, \lambda} f \right)(x) = 0 \quad \text{for almost every } x \in \mathbb{R}.$$

Apply $\mathbb{D}_-^{H-\frac{1}{2}, \lambda}$ to both sides of equation (28) and use Lemma 6.6 to get $f(x) = 0$ for almost every $x \in \mathbb{R}$, and hence \mathcal{A}_1 is an inner product space.

Next, we want to show that the set of elementary functions \mathcal{E} is dense in \mathcal{A}_1 . For any $f \in \mathcal{A}_1$, we also have $f \in L^2(\mathbb{R})$, and hence there exists a sequence of elementary functions (f_n) in $L^2(\mathbb{R})$ such that $\|f - f_n\|_2 \rightarrow 0$. But

$$\|f - f_n\|_{\mathcal{A}_1} = \langle f - f_n, f - f_n \rangle_{\mathcal{A}_1} = \langle F - F_n, F - F_n \rangle_2 = \|F - F_n\|_2,$$

where $F_n(x) = \left(\mathbb{I}_-^{H-\frac{1}{2},\lambda} f_n\right)(x)$ and $F(x)$ is given by (27). Lemma 6.2 implies that

$$\|f - f_n\|_{\mathcal{A}_1} = \|F - F_n\|_2 = \|\mathbb{I}_-^{H-\frac{1}{2},\lambda}(f - f_n)\|_2 \leq C\|f - f_n\|_2$$

for some $C > 0$, and since $\|f - f_n\|_2 \rightarrow 0$, it follows that the set of elementary functions is dense in \mathcal{A}_1 .

Finally, we provide an example to show that \mathcal{A}_1 is not complete. The functions

$$\widehat{f}_n(\omega) = |\omega|^{-p} \mathbf{1}_{\{1 < |\omega| < n\}}(\omega), \quad p > 0,$$

are in $L^2(\mathbb{R})$, $\overline{\widehat{f}_n(\omega)} = \widehat{f}_n(-\omega)$, and hence they are the Fourier transforms of functions $f_n \in L^2(\mathbb{R})$. Apply Lemma 6.3 to see that the corresponding functions $F_n(x) = \Gamma(H - \frac{1}{2}) \left(\mathbb{I}_-^{H-\frac{1}{2},\lambda} f_n\right)(x)$ from (27) have Fourier transform

$$(29) \quad \mathcal{F}[F_n](\omega) = \Gamma(H - \frac{1}{2})(\lambda + i\omega)^{\frac{1}{2}-H} \widehat{f}_n(\omega).$$

Since $\frac{1}{2} - H < 0$, it follows that

$$\|F_n\|_2^2 = \|\widehat{F}_n\|_2^2 = \Gamma(H - \frac{1}{2})^2 \int_{-\infty}^{\infty} \left| \widehat{f}_n(\omega) \right|^2 (\lambda^2 + \omega^2)^{\frac{1}{2}-H} < \infty$$

for each n , which shows that $f_n \in \mathcal{A}_1$. Now it is easy to check that $f_n - f_m \rightarrow 0$ in \mathcal{A}_1 , as $n, m \rightarrow \infty$, whenever $p > 1 - H$, so that (f_n) is a Cauchy sequence. Choose $p = \frac{1}{2}$ and suppose that there exists some $f \in \mathcal{A}_1$ such that $\|f_n - f\|_{\mathcal{A}_1} \rightarrow 0$ as $n \rightarrow \infty$. Then

$$(30) \quad \int_{-\infty}^{\infty} \left| \widehat{f}_n(\omega) - \widehat{f}(\omega) \right|^2 (\lambda^2 + \omega^2)^{\frac{1}{2}-H} \rightarrow 0$$

as $n \rightarrow \infty$, and since, for any given $m \geq 1$, the value of $\widehat{f}_n(\omega)$ does not vary with $n > m$ whenever $\omega \in [-m, m]$, it follows that $\widehat{f}(\omega) = |\omega|^{-\frac{1}{2}} \mathbf{1}_{\{|\omega| > 1\}}$ on any such interval. Since m is arbitrary, it follows that $\widehat{f}(\omega) = |\omega|^{-\frac{1}{2}} \mathbf{1}_{\{|\omega| > 1\}}$, but this function is not in $L^2(\mathbb{R})$, so $\widehat{f}(\omega) \notin \mathcal{A}_1$, which is a contradiction. Hence \mathcal{A}_1 is not complete, and this completes the proof. \square

Remark 3.3. It follows from Lemma 6.2 that \mathcal{A}_1 contains every function in $L^2(\mathbb{R})$, and hence they are the same set, but endowed with a different inner product.

We now define the stochastic integral with respect to THP^1 for any function in \mathcal{A}_1 in the case where $\frac{1}{2} < H < 1$.

Definition 3.4. For any $\frac{1}{2} < H < 1$ and $\lambda > 0$, we define

$$(31) \quad \int_{\mathbb{R}} f(x) Z_{H,\lambda}^1(dx) := \Gamma(H - \frac{1}{2}) \int_{\mathbb{R}} \left(\mathbb{I}_-^{H-\frac{1}{2},\lambda} f\right)(x) B(dx)$$

for any $f \in \mathcal{A}_1$.

Theorem 3.5. *For any $\frac{1}{2} < H < 1$ and $\lambda > 0$, the stochastic integral $\mathcal{I}^{\alpha,\lambda}$ in (31) is an isometry from \mathcal{A}_1 into $\overline{\text{Sp}}(Z_{H,\lambda}^1)$. Since \mathcal{A}_1 is not complete, these two spaces are not isometric.*

Proof. It follows from Lemma 6.2 that the stochastic integral (31) is well-defined for any $f \in \mathcal{A}_1$. Proposition 2.1 in Pipiras and Taqqu [29] implies that, if \mathcal{D} is an inner product space such that $(f, g)_{\mathcal{D}} = \langle \mathcal{I}^{\alpha,\lambda}(f), \mathcal{I}^{\alpha,\lambda}(g) \rangle_{L^2(\Omega)}$ for all $f, g \in \mathcal{E}$, and if \mathcal{E} is dense \mathcal{D} , then there is an isometry between \mathcal{D} and a linear subspace of $\overline{\text{Sp}}(Z_{H,\lambda}^1)$ that extends the map $f \rightarrow \mathcal{I}^{\alpha,\lambda}(f)$ for $f \in \mathcal{E}$, and furthermore, \mathcal{D} is isometric to $\overline{\text{Sp}}(Z_{H,\lambda}^1)$ itself if and only if \mathcal{D} is complete. Using the Itô isometry and the definition (31), it follows from (26) that for any $f, g \in \mathcal{A}_1$ we have

$$\langle f, g \rangle_{\mathcal{A}_1} = \langle F, G \rangle_{L^2(\mathbb{R})} = \langle \mathcal{I}^{\alpha,\lambda}(f), \mathcal{I}^{\alpha,\lambda}(g) \rangle_{L^2(\Omega)},$$

and then the result follows from Theorem 3.2. \square

We now apply the spectral domain representation of THP^1 to investigate the stochastic integral with respect to THP^1 . Apply the Fourier transform of an indicator function to write this spectral domain representation in the form

$$Z_{H,\lambda}^1(t) = \Gamma(H - \frac{1}{2}) \int_{-\infty}^{+\infty} \widehat{\mathbf{1}}_{[0,t]}(\omega) (\lambda + i\omega)^{\frac{1}{2}-H} \hat{B}(d\omega).$$

It follows easily that for any elementary function (20) we may write

$$(32) \quad \mathcal{I}^{\alpha,\lambda}(f) = \Gamma(H - \frac{1}{2}) \int_{-\infty}^{\infty} \widehat{f}(\omega) (\lambda + i\omega)^{\frac{1}{2}-H} \hat{B}(d\omega),$$

and then for any elementary functions f and g we have

$$(33) \quad \langle \mathcal{I}^{\alpha,\lambda}(f), \mathcal{I}^{\alpha,\lambda}(g) \rangle_{L^2(\Omega)} = \Gamma(H - \frac{1}{2}) \int_{-\infty}^{\infty} \widehat{f}(\omega) \overline{\widehat{g}(\omega)} (\lambda^2 + \omega^2)^{\frac{1}{2}-H} d\omega.$$

Theorem 3.6. *For any $\frac{1}{2} < H < 1$ and $\lambda > 0$, the class of functions*

$$(34) \quad \mathcal{A}_2 := \left\{ f \in L^2(\mathbb{R}) : \int \left| \widehat{f}(\omega) \right|^2 (\lambda^2 + \omega^2)^{\frac{1}{2}-H} d\omega < \infty \right\},$$

is a linear space with the inner product

$$(35) \quad \langle f, g \rangle_{\mathcal{A}_2} = \Gamma(H - \frac{1}{2})^2 \int_{-\infty}^{+\infty} \widehat{f}(\omega) \overline{\widehat{g}(\omega)} (\lambda^2 + \omega^2)^{\frac{1}{2}-H} d\omega.$$

The set of elementary functions \mathcal{E} is dense in the space \mathcal{A}_2 . The space \mathcal{A}_2 is not complete.

Proof. The proof combines Theorem 3.2 and using the Plancherel Theorem.

Since $H > \frac{1}{2}$, the function $(\lambda^2 + \omega^2)^{\frac{1}{2}-H}$ is bounded by a constant $C(H, \lambda)$ that depends only on H and λ , so for any $f \in L^2(\mathbb{R})$ we have

$$(36) \quad \int_{\mathbb{R}} |\widehat{f}(\omega)|^2 (\lambda^2 + \omega^2)^{\frac{1}{2}-H} d\omega \leq C(H, \lambda) \int_{\mathbb{R}} |\widehat{f}(\omega)|^2 d\omega < \infty$$

and hence $f \in \mathcal{A}_2$. Since $\mathcal{A}_2 \subset L^2(\mathbb{R})$ by definition, this proves that $L^2(\mathbb{R})$ and \mathcal{A}_2 are the same set of functions, and then it follows from Lemma 6.2 that \mathcal{A}_1 and \mathcal{A}_2 are the same set of functions. Observe that $\varphi_f = \left(\mathbb{I}_-^{H-\frac{1}{2}, \lambda} f \right)$ is again a function with Fourier transform

$$\widehat{\varphi}_f = (\lambda + i\omega)^{\frac{1}{2}-H} \widehat{f}.$$

Then it follows from the Plancherel Theorem that

$$\begin{aligned} \langle f, g \rangle_{\mathcal{A}_1} &= \Gamma(H - \frac{1}{2})^2 \langle \varphi_f, \varphi_g \rangle_2 = \Gamma(1 - \alpha)^2 \langle \widehat{\varphi}_f, \widehat{\varphi}_g \rangle_2 \\ &= \Gamma(1 - \alpha)^2 \int_{-\infty}^{+\infty} \widehat{f}(\omega) \overline{\widehat{g}(\omega)} (\lambda^2 + \omega^2)^{\frac{1}{2}-H} d\omega = \langle f, g \rangle_{\mathcal{A}_2} \end{aligned}$$

and hence the two inner products are identical. Then the conclusions of Theorem 3.6 follow from Theorem 3.2. \square

Definition 3.7. For any $H > \frac{1}{2}$ and $\lambda > 0$, we define

$$(37) \quad \mathcal{I}^{\alpha, \lambda}(f) = \Gamma(1 - \alpha) \int_{-\infty}^{\infty} \widehat{f}(\omega) (\lambda + i\omega)^{\frac{1}{2}-H} \widehat{B}(d\omega)$$

for any $f \in \mathcal{A}_2$.

Theorem 3.8. For any $H > \frac{1}{2}$ and $\lambda > 0$, the stochastic integral $\mathcal{I}^{\alpha, \lambda}$ in (37) is an isometry from \mathcal{A}_2 into $\overline{\text{Sp}}(Z_{H, \lambda}^1)$. Since \mathcal{A}_2 is not complete, these two spaces are not isometric.

Proof. The proof of Theorem 3.6 shows that \mathcal{A}_1 and \mathcal{A}_2 are identical when $H > \frac{1}{2}$. Then the result follows immediately from Theorems 3.5. \square

Now, we consider the second case that we mentioned at the beginning of this section. we will show that $Z_{H, \lambda}^1$ is a continuous semimartingale with finite variation and hence one can define stochastic integrals $I(f) := \int f(x) Z_{H, \lambda}^1(dx)$ in the standard manner, via the Itô stochastic calculus (e.g., see Kallenberg [15, Chapter 15]).

Theorem 3.9. A tempered Hermite process of order one $\{Z_{H, \lambda}^1(t)\}_{t \geq 0}$ with $H > 1$ and $\lambda > 0$ is a continuous semimartingale with the canonical decomposition

$$(38) \quad Z_{H, \lambda}^1(t) = \int_0^t M_{H, \lambda}(s) ds$$

where

$$(39) \quad M_{H, \lambda}(s) := \int_{-\infty}^{+\infty} (s - y)_+^{H-\frac{3}{2}} e^{-\lambda(s-y)_+} B(dy).$$

Moreover, $\{Z_{H,\lambda}^1(t)\}_{t \geq 0}$ is a finite variation process.

Proof. Let $\{\mathcal{F}_t^B\}_{t \geq 0}$ be the σ -algebra generated by $\{B_s : 0 \leq s \leq t\}$. Given a function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g(t) = 0$ for all $t < 0$, and

$$(40) \quad g(t) = C + \int_0^t h(s) ds, \quad \text{for all } t > 0,$$

for some $C \in \mathbb{R}$ and some $h \in L^2(\mathbb{R})$, a result of Cheridito [7, Theorem 3.9] shows that the Gaussian stationary increment process

$$(41) \quad Y_t^g := \int_{\mathbb{R}} [g(t-y) - g(-y)] B(dy), \quad t \geq 0$$

is a continuous $\{\mathcal{F}_t^B\}_{t \geq 0}$ semimartingale with canonical decomposition

$$(42) \quad Y_t^g = g(0)B_t + \int_0^t \int_{-\infty}^s h(s-y)B(dy)ds,$$

and conversely, that if (41) defines a semimartingale on $[0, T]$ for some $T > 0$, then g satisfies these properties. Define $g(t) = 0$ for $t \leq 0$ and

$$(43) \quad g(t) := \int_0^t s^{H-\frac{3}{2}} e^{-\lambda s} ds \quad \text{for } t > 0.$$

It is easy to check that the function $g(t-y) - g(-y)$ is square integrable over the entire real line for any $H > \frac{1}{2}$ and $\lambda > 0$ (See Lemma 2.2). Next observe that (40) holds with $C = 0$, $h(s) = 0$ for $s < 0$ and

$$(44) \quad h(s) := s^{H-\frac{3}{2}} e^{-\lambda s} \in L^2(\mathbb{R})$$

for any $H > 1$ and $\lambda > 0$. Then it follows from [7, Theorem 3.9] that THP^1 is a continuous semimartingale with canonical decomposition

$$(45) \quad \begin{aligned} Z_{H,\lambda}^1 &= \int_{-\infty}^{+\infty} \int_0^t (s-y)_+^{H-\frac{3}{2}} e^{-\lambda(s-y)_+} ds B(dy) \\ &= \int_0^t \int_{-\infty}^{+\infty} (s-y)_+^{H-\frac{3}{2}} e^{-\lambda(s-y)_+} B(dy) ds \end{aligned}$$

which reduces to (38). Since $C = 0$, Theorem 3.9 in [7] implies that $\{Z_{H,\lambda}^1(t)\}$ is a finite variation process. \square

Remark 3.10. When $H = \frac{3}{2}$ and $\lambda > 0$, the Gaussian stochastic process (39) is an Ornstein-Uhlenbeck process. When $H > 1$ and $\lambda > 0$, it is a one dimensional Matérn stochastic process [4, 10, 13], also called a “fractional Ornstein-Uhlenbeck process” in the physics literature [19]. It follows from Knight [17, Theorem 6.5] that $M_{H,\lambda}(t)$ is a semimartingale in both cases.

Cheridito [7, Theorem 3.9] provides a necessary and sufficient condition for the process (41) to be a semimartingale, and then it is not hard to check that THP^1 is *not a semimartingale* in the remaining case when $\frac{1}{2} < H < 1$.

4. ARTFIMA TIME SERIES; DEFINITION AND BASIC PROPERTIES

In this section, we first recall the definition of the autoregressive tempered fractionally integrated moving average (ARTFIMA) time series and some of its basic properties such as the covariance function and spectral density.

The tempered fractional difference operator is defined by:

$$(46) \quad \Delta_h^{\alpha, \lambda} f(x) = \sum_{j=0}^{\infty} w_j e^{-\lambda j h} f(x - j h) \quad \text{with} \quad w_j := (-1)^j \binom{\alpha}{j} = \frac{(-1)^j \Gamma(1 + \alpha)}{j! \Gamma(1 + \alpha - j)}$$

for $\alpha > 0$ and $\lambda > 0$, where $\Gamma(\cdot)$ is the Euler gamma function. If $\lambda = 0$ and α is a positive integer, then equation (46) reduces to the usual definition of the fractional difference operator.

The ARMA(p, q) model, which combines an autoregression of order p with a moving average of order q , is defined by

$$(47) \quad X_t - \sum_{j=1}^p \phi_j X_{t-j} = Z_t + \sum_{i=1}^q \theta_i Z_{t-i}$$

where $\{Z_t\}$ is an i.i.d. sequence of uncorrelated random variables (white noise). We now recall the definition of the ARTFIMA (p, α, λ, q).

Definition 4.1. *The discrete time stochastic process $\{X_t\}$ is called an autoregressive tempered fractional integrated moving average, ARTFIMA (p, λ, α, q), if*

$$(48) \quad \Delta_1^{\alpha, \lambda} X_t = \sum_{j=0}^{\infty} w_j e^{-\lambda j h} X_{t-j},$$

follows an ARMA(p, q) model.

Let $\{X_t\}$ be an ARTFIMA ($0, \lambda, \alpha, 0$) process. Then,

$$X_t = \Delta_1^{-\alpha, \lambda} Z_t = \sum_{j=0}^{\infty} (-1)^j e^{-\lambda j} \binom{-\alpha}{j} Z_{t-j},$$

where $\Delta_1^{-\alpha, \lambda}$ is the inverse operator of $\Delta_1^{\alpha, \lambda}$ and can be defined by (46). In other words, $X_t = \Delta_1^{-\alpha, \lambda} Z_t$, is a tempered fractionally integrated ARMA(p, q) model. The fractional integration operator $\Delta_1^{-\alpha, \lambda}$, the inverse of $\Delta_1^{\alpha, \lambda}$, is also defined by (46). We refer the reader to [31] to find more details about the tempered fractional difference operator. In this paper we are interested in the case ARTFIMA ($0, \alpha, \lambda, 0$).

Remark 4.2. Since $\{Z_t\}$ is stationary and

$$\sum_{j=0}^{\infty} (-1)^j e^{-\lambda j} \binom{-\alpha}{j} = (1 - e^{-\lambda})^{-\alpha} < \infty$$

for any $\alpha > 0$ and $\lambda > 0$, Proposition 3.1.2 in [5] implies that the series

$$X_t = \Delta_1^{-\alpha, \lambda} Z_t = \sum_{j=0}^{\infty} (-1)^j e^{-\lambda j} \binom{-\alpha}{j} Z_{t-j}$$

is stationary and converges absolutely with probability one.

Remark 4.3. Peiris [27] has proposed a generalized autoregressive GAR(p) time series model $(1 - \beta B)^\alpha X_t = Z_t$ for applications in finance, where $|\beta| < 1$. Taking $\beta = e^{-\lambda}$ we obtain the ARTFIMA($0, \alpha, \lambda, 0$) model.

We next state the spectral density and covariance function of ARTFIMA ($0, \alpha, \lambda, 0$).

Theorem 4.4. *Let $\{X_t\}$ be an ARTFIMA ($0, \alpha, \lambda, 0$) times series.*

a: $\{X_t\}$ has the spectral density

$$(49) \quad h(\omega) = \frac{\sigma^2}{2\pi} \left| 1 - e^{-(\lambda + i\omega)} \right|^{-2\alpha},$$

for $-\pi \leq \omega \leq \pi$.

b: The covariance function of $\{X_t\}$ is

$$(50) \quad \gamma_k = \mathbb{E}(X_t X_{t+k}) = \frac{\sigma^2 e^{-\lambda k} \Gamma(\alpha + k)}{2\pi \Gamma(\alpha) k!} {}_2F_1(\alpha; k + \alpha; k + 1; e^{-2\lambda}),$$

where ${}_2F_1(a; b; c; z) = \sum_{j=0}^{\infty} \frac{\Gamma(a+j)\Gamma(b+j)\Gamma(c)z^j}{\Gamma(a)\Gamma(b)\Gamma(c+j)\Gamma(j+1)}$ is the hypergeometric function.

Proof. (a) Writing $X_t = \psi_\lambda(B)Z_t$, we have $\psi_\lambda(B) = (1 - e^{-\lambda}B)^{-\alpha}$. Then the general theory of linear filters implies that X_t has spectral density $f_X(k) = |\Psi(e^{-ik})|^2 f_Z(k)$ using the complex absolute value (e.g., see [5]). That is

$$\begin{aligned} h(\omega) &= \frac{\sigma^2}{2\pi} \psi_\lambda(e^{i\omega}) \psi_\lambda(e^{-i\omega}) \\ &= \frac{\sigma^2}{2\pi} (1 - 2e^{-\lambda} \cos \omega + e^{-2\lambda})^{-\alpha} \\ &= \left| 1 - e^{-(\lambda + i\omega)} \right|^{-2\alpha} \end{aligned}$$

and this gives (49). In order to show (b), we have

$$\begin{aligned}
\gamma_k &= \int_{-\pi}^{\pi} \cos(k\omega) h(\omega) d\omega \\
&= \int_{-\pi}^{\pi} \frac{\sigma^2}{2\pi} \frac{\cos(k\omega)}{(1 - 2e^{-\lambda} \cos \omega + e^{-2\lambda})^\alpha} d\omega \\
&= \int_0^{2\pi} \frac{\sigma^2}{2\pi} \frac{(-1)^k \cos(k\omega')}{(1 - 2e^{-\lambda} \cos \omega + e^{-2\lambda})^\alpha} d\omega' [\omega' := \omega + \pi] \\
&= \sigma^2 \frac{e^{-\lambda k} \Gamma(k + \alpha)}{\Gamma(\alpha) k!} {}_2F_1(\alpha; k + \alpha; k + 1; e^{-2\lambda}),
\end{aligned}$$

where we applied the following integral formula:

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\cos k\omega'}{(1 - 2z \cos \omega + z^2)^\alpha} d\omega' = \frac{z^k \Gamma(k - \alpha)}{\Gamma(\alpha) k!} {}_2F_1(\alpha; k + \alpha; k + 1; z^2)$$

and hence we proved part (b). \square

The next lemma gives the spectral representation of the ARTFIMA $(0, \alpha, \lambda, 0)$. We will use this lemma in the next section.

Lemma 4.5. *Let X_k^λ be the ARTFIMA $(0, \alpha, \lambda, 0)$ time series such that $X_k^\lambda = \sum_{j=0}^{\infty} (-1)^j e^{-\lambda j} \binom{-\alpha}{j} Z_{t-j}$. Then, X_k^λ has the spectral representation*

$$X_k^\lambda = \int_{-\pi}^{\pi} e^{ik\nu} dW_\lambda(\nu),$$

where $dW_\lambda(\nu) = \left(1 - e^{-(\lambda + i\nu)}\right)^{-\alpha} dW$ and $\{W(\nu), -\pi \leq \nu \leq \pi\}$ is a right-continuous orthogonal increment process.

Proof. Suppose $\{Z_t\}$ has the spectral representation $Z_t = \int_{-\pi}^{\pi} e^{it\nu} dW(\nu)$, where $\{W(\nu), -\pi \leq \nu \leq \pi\}$ is a right-continuous orthogonal increment process. Then Theorem 4.10.1 in [5] implies that X_k^λ has the spectral representation

$$\begin{aligned}
X_k^\lambda &= \int_{-\pi}^{\pi} e^{ik\nu} \sum_{j=0}^{\infty} (-1)^j e^{-\lambda j} \binom{-\alpha}{j} e^{-ij\nu} dW(\nu) \\
&= \int_{-\pi}^{\pi} e^{ik\nu} \left(1 - e^{-(\lambda + i\nu)}\right)^{-\alpha} dW(\nu),
\end{aligned}$$

and this completes the proof. \square

The next lemma gives the asymptotic result of the covariance function of the ARTFIMA $(0, \alpha, \lambda, 0)$.

Lemma 4.6. *Let $\{X_t\}$ be an ARTFIMA $(0, \alpha, \lambda, 0)$ times series with the covariance function $\gamma_k = \mathbb{E}(X_t X_{t+k})$ given by (50). Then*

$$\gamma_k \sim \frac{\sigma^2}{2\pi} \frac{1}{\Gamma(\alpha)} e^{-\lambda k} k^{\alpha-1} (1 - e^{-2\lambda})^{-\alpha}$$

as $|k| \rightarrow \infty$.

Proof. By applying (50) and the fact that $\frac{\Gamma(a+k)}{\Gamma(b+k)} \sim k^{a-b}$, as $k \rightarrow \infty$, we have

$$\begin{aligned} \gamma_k &= \sigma^2 \frac{e^{-\lambda k} \Gamma(k + \alpha)}{\Gamma(\alpha) k!} {}_2F_1(\alpha; k + \alpha; k + 1; e^{-2\lambda}) \\ &= \sigma^2 \frac{e^{-\lambda k} \Gamma(k + \alpha)}{\Gamma(\alpha) k!} \sum_{j=0}^{\infty} \frac{\Gamma(\alpha + j) \Gamma(k + \alpha + j) \Gamma(k + 1) e^{-2\lambda j}}{\Gamma(\alpha) \Gamma(k + \alpha) \Gamma(k + j + 1) (j)!} \\ &\sim \sigma^2 \frac{1}{\Gamma(\alpha)} e^{-\lambda k} k^{\alpha-1} \sum_{j=0}^{\infty} \frac{\Gamma(\alpha + j) e^{-2\lambda j}}{\Gamma(\alpha) (j)!} \\ &= \sigma^2 \frac{1}{\Gamma(\alpha)} e^{-\lambda k} k^{\alpha-1} (1 - e^{-2\lambda})^{-\alpha}, \end{aligned}$$

which gives the desired result. \square

Remark 4.7. The ARTFIMA $(0, \alpha, \lambda, 0)$ is short memory process, since by Lemma 4.6 one can show that $\sum_{k=0}^{\infty} \gamma_k < \infty$.

5. WEAK CONVERGENCE RESULTS

We now in a position to answer the first and second question. We start with the first one.

Assume $H = \frac{1}{2} + \alpha$ for $\alpha > 0$ and let $\{Z_j\}_{j \in \mathbb{Z}}$ be a sequence of independent and identically distributed random variables mean zero and variance one. Define the random variables

$$(51) \quad Y_k^{\frac{\lambda}{n}} := \sum_{j \in \mathbb{Z}} C_j^{\frac{\lambda}{n}} Z_{k-j}, \quad k = 1, 2, \dots$$

where

$$(52) \quad C_j^{\frac{\lambda}{n}} = \begin{cases} \frac{1}{\Gamma(\alpha)} j^{\alpha-1} e^{-\frac{\lambda}{n} j} & \text{if } j \geq 1 \\ 0 & \text{if } j \leq 0. \end{cases}$$

For $t \geq 0$, we define $S_n^{\frac{\lambda}{n}}(t)$ as the partial sum of $\{X_k^{\frac{\lambda}{n}}\}$,

$$(53) \quad S_n^{\frac{\lambda}{n}}(t) := \sum_{k=1}^{[t]} Y_k^{\frac{\lambda}{n}} + (t - [t]) Y_{[t]+1}^{\frac{\lambda}{n}}, \quad t \geq 0,$$

where $[t]$ is the largest integer less than or equals t and $\sum_{k=1}^0 = 0$. We also define

$$(54) \quad \xi_m^{\frac{\lambda}{n}}(t) := \sum_{j=1-m}^{[t]-m} C_j^{\frac{\lambda}{n}} + (t - [t])C_{t+1-m}^{\frac{\lambda}{n}},$$

for $m \in \mathbb{Z}$ and $t \geq 0$, where $\sum_{j=1-m}^{-m} = 0$. Then we have from (51) and (53),

$$(55) \quad S^{\frac{\lambda}{n}}(t) = \sum_{m \in \mathbb{Z}} \xi_m^{\frac{\lambda}{n}}(t) Z_m.$$

On the other hand,

$$(56) \quad \begin{aligned} \xi_m^{\frac{\lambda}{n}}(nt) &= \sum_{j=1-m}^{[nt]-m} C_j^{\frac{\lambda}{n}} \sim \int_{-m}^{[nt]-m} j^{\alpha-1} e^{-\frac{\lambda}{n}j} dj \\ &= \left(\frac{n}{\lambda}\right)^{\alpha} \int_{-\frac{\lambda m}{n}}^{\frac{\lambda}{n}([nt]-m)} \omega^{\alpha-1} e^{-\omega} d\omega \\ &= \left(\frac{n}{\lambda}\right)^{\alpha} \left[\gamma\left(\alpha, \frac{\lambda}{n}([nt] - m)\right) - \gamma\left(-\frac{\lambda m}{n}\right) \right], \end{aligned}$$

when m is negative and $|m|$ is large. From (55) and (56) we have:

Lemma 5.1. *For any $\theta_1, \theta_2, \dots, \theta_p, t_1, t_2, \dots, t_p \geq 0$, we have*

$$n^{-2H} \sum_{m \in \mathbb{Z}} \left| \sum_{r=1}^p \theta_r \xi_m^{\frac{\lambda}{n}}(nt) \right|^2 \rightarrow \int_{-\infty}^{+\infty} \left| \sum_{r=1}^p \theta_r \int_0^t (s-y)_+^{H-\frac{3}{2}} e^{-\lambda(s-y)_+} ds \right|^2 dy$$

as $n \rightarrow \infty$.

Proof. By applying (56), we get:

$$\begin{aligned}
& n^{-2H} \sum_{m \in \mathbb{Z}} \left| \sum_{r=1}^p \theta_r \xi_m^{\frac{\lambda}{n}}(nt_r) \right|^2 \\
& \sim n^{-2H} \left(\frac{n}{\lambda} \right)^{2\alpha} \sum_{m \in \mathbb{Z}} \left| \sum_{r=1}^p \theta_r \left[\gamma\left(\alpha, \frac{\lambda}{n}([nt] - m)\right) - \gamma\left(\alpha, -\frac{\lambda m}{n}\right) \right] \right|^2 \\
& = \left(\frac{1}{\lambda} \right)^{2\alpha} n^{2\alpha-2H+1} n^{-1} \sum_{m \in \mathbb{Z}} \left| \sum_{r=1}^p \theta_r \left[\gamma\left(\alpha, \frac{\lambda}{n}([nt] - m)\right) - \gamma\left(-\frac{\lambda m}{n}\right) \right] \right|^2 \\
& \rightarrow \left(\frac{1}{\lambda} \right)^{2\alpha} \int_{\mathbb{R}} \left| \sum_{r=1}^p \theta_r \int_{-\lambda y}^{\lambda t_r - \lambda y} x_+^{\alpha-1} e^{-(x)_+} dx \right|^2 dy, \quad (\text{as } n \rightarrow \infty), \\
& \quad (\text{define } \lambda s = x + \lambda y) \\
& = \int_{\mathbb{R}} \left| \sum_{r=1}^p \theta_r \int_0^{t_r} (s-y)_+^{\alpha-1} e^{-\lambda(s-y)_+} ds \right|^2 dy \\
& = \int_{\mathbb{R}} \left| \sum_{r=1}^p \theta_r \int_0^{t_r} (s-y)_+^{H-\frac{3}{2}} e^{-\lambda(s-y)_+} ds \right|^2 dy
\end{aligned}$$

and this completes the proof. \square

Theorem 5.2. *The finite dimensional distribution of $n^{-H} S_n^{\frac{\lambda}{n}}(nt)$ converge in distribution to $Z_{H,\lambda}^1(t)$, given by (1), as $n \rightarrow \infty$. That is*

$$\left(n^{-H} S_n^{\frac{\lambda}{n}}(nt_1), n^{-H} S_n^{\frac{\lambda}{n}}(nt_2), \dots, n^{-H} S_n^{\frac{\lambda}{n}}(nt_p) \right) \rightarrow \left(Z_{H,\lambda}^1(t_1), Z_{H,\lambda}^1(t_2), \dots, Z_{H,\lambda}^1(t_p) \right)$$

as $n \rightarrow \infty$.

Proof. Let $\theta_1, \theta_2, \dots, \theta_p, t_1, t_2, \dots, t_p \geq 0$. Then by computing the characteristic function of $n^{-H} \sum_{r=1}^p \theta_r S_n^{\frac{\lambda}{n}}(nt_r)$ we get

$$\begin{aligned}
(57) \quad \mathbb{E} \left[\exp \left\{ i n^{-H} \sum_{r=1}^p \theta_r S_n^{\frac{\lambda}{n}}(nt_r) \right\} \right] &= \mathbb{E} \left[\exp \left\{ i n^{-H} \sum_{m \in \mathbb{Z}} \sum_{r=1}^p \theta_r \xi_m^{\frac{\lambda}{n}}(nt_r) Z_m \right\} \right] \\
&= \prod_{m \in \mathbb{Z}} \left[\exp \left\{ i n^{-H} \sum_{r=1}^p \theta_r \xi_m^{\frac{\lambda}{n}}(nt_r) Z_m \right\} \right] \\
&= \prod_{m \in \mathbb{Z}} \exp \left\{ -n^{-2H} \left| \sum_{r=1}^p \theta_r \xi_m^{\frac{\lambda}{n}}(nt_r) \right|^2 \right\} \\
&= \exp \left\{ - \sum_{m \in \mathbb{Z}} n^{-2H} \left| \sum_{r=1}^p \theta_r \xi_m^{\frac{\lambda}{n}}(nt_r) \right|^2 \right\}.
\end{aligned}$$

Taking the limit of (57) yields

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mathbb{E} \left[\exp \left\{ i n^{-H} \sum_{r=1}^p \theta_r S_n^{\frac{\lambda}{n}}(nt_r) \right\} \right] &= \exp \left[- \lim_{n \rightarrow \infty} n^{-2H} \sum_{m \in \mathbb{Z}} \left| \sum_{r=1}^p \theta_r \xi_m^{\frac{\lambda}{n}}(nt_r) \right|^2 \right] \\
&= \exp \left\{ \int_{\mathbb{R}} \left| \sum_{r=1}^p \theta_r \int_0^{t_r} (s-y)_+^{H-\frac{3}{2}} e^{-\lambda(s-y)_+} ds \right|^2 dy \right\} \\
&= \mathbb{E} \left[\exp \left\{ i \sum_{r=1}^p \theta_r Z_{H,\lambda}^1(t_r) \right\} \right]
\end{aligned}$$

as $n \rightarrow \infty$ and this completes the proof. \square

Theorem 5.3. *Let $\{Z_j\}_{j \in \mathbb{Z}}$ be a sequence of i.i.d random variables with mean zero and finite variance. Then $n^{-H} S_n^{\frac{\lambda}{n}}(nt)$ converges weakly to $Z_{H,\lambda}^1(t)$, given by (1), in $C[0, 1]$, as $n \rightarrow \infty$ ($C[0, 1]$ is the space of all continuous functions defined on $[0, 1]$). That is*

$$(58) \quad n^{-H} S_n^{\frac{\lambda}{n}}(nt) \Rightarrow Z_{H,\lambda}^1(t),$$

where \Rightarrow means weak convergence in $C[0, 1]$.

Proof. In Theorem 5.2, We have shown the finite dimensional convergence of $n^{-H} S_n^{\frac{\lambda}{n}}(nt)$ to $Z_{H,\lambda}^1(t)$. Therefore, here, we just need to prove the tightness of $n^{-H} S_n^{\frac{\lambda}{n}}(nt)$. We show that for $0 \leq t_1 \leq t_2 \leq 1$,

$$(59) \quad \mathbb{E} \left[\left| n^{-H} S_n^{\frac{\lambda}{n}}(nt_2) - n^{-H} S_n^{\frac{\lambda}{n}}(nt_1) \right|^2 \right] \leq C |t_2 - t_1|^{2H},$$

where C is a constant. First apply (56) to get

$$\begin{aligned}
(60) \quad & \sum_{m \in \mathbb{Z}} |\xi_m^{\frac{\lambda}{n}}(nt_2) - \xi_m^{\frac{\lambda}{n}}(nt_1)| \\
& \leq \left(\frac{n}{\lambda} \right)^{2\alpha} \int_{\mathbb{R}} \left| \int_{\frac{\lambda}{n}([nt_1]-x)}^{\frac{\lambda}{n}([nt_2]-x)} \omega_+^{\alpha-1} e^{-(\omega)_+} d\omega \right|^2 dx \\
& = \int_{\mathbb{R}} \left| \int_{[nt_1]-x}^{[nt_2]-x} y_+^{\alpha-1} e^{-\frac{\lambda}{n}(y)_+} dy \right|^2 dx \quad (y := \frac{n\omega}{\lambda}) \\
& = \int_{\mathbb{R}} \left| \int_0^{[nt_2]-[nt_1]} \left(z + (ns-x) \right)_+^{\alpha-1} e^{-\frac{\lambda}{n}(z+(ns-x))_+} dz \right|^2 dx \quad (z := y - (ns-x)) \\
& \leq C \int_{\mathbb{R}} \left| (nt_2-x)_+^\alpha - (nt_1-x)_+^\alpha \right|^2 dx.
\end{aligned}$$

Maejima [20] proved that

$$(61) \quad \int_{\mathbb{R}} \left| |nt_2 - x|^\alpha - |nt_1 - x|^\alpha \right|^2 \leq n^{1+2\alpha} (t_2 - t_1)^{1+2\alpha}.$$

Hence by applying (60) and (61) we have

$$\begin{aligned} & \mathbb{E} \left[\left| n^{-H} \left(S_n^\lambda(nt_2) - S_n^\lambda(nt_1) \right) \right|^2 \right] \\ &= n^{-2H} \mathbb{E} \left(\sum_{m \in \mathbb{Z}} \left| \left(\xi_m^\lambda(nt_2) - \xi_m^\lambda(nt_1) \right) Z_m \right|^2 \right) \\ &= n^{-2H} \sigma^2 \sum_{m \in \mathbb{Z}} \left| \xi_m^\lambda(nt_2) - \xi_m^\lambda(nt_1) \right|^2 \\ &\leq C \sigma^2 n^{-2H} n^{1+2\alpha} |t_2 - t_1|^{1+2\alpha} \quad \left(H = \alpha + \frac{1}{2} \right) \\ &\leq C |t_2 - t_1|^{2H}. \end{aligned}$$

Thus the tightness of $n^{-H} S_n^\lambda(nt)$ follows from Theorem 12.3 in [3] and this completes the proof. \square

Theorem 5.4. *Let $\alpha > 0$ and X_k^λ be the ARTFIMA $(0, \alpha, \lambda, 0)$. Suppose*

$$T_n^\lambda(t) := \sum_{k=1}^{[t]} X_k^\lambda + (t - [t]) X_{[t]+1}^\lambda, \quad t \geq 0.$$

Then,

$$n^{-H} T_n^\lambda(nt) \Rightarrow Z_{H,\lambda}^1(t)$$

as $n \rightarrow \infty$ in $C[0, 1]$.

Proof. It follows from Stirling's approximation that

$$\omega_j^{\frac{\lambda}{n}} = (-1)^j \binom{-\alpha}{j} e^{-\frac{\lambda}{n}} \sim \frac{\alpha}{\Gamma(1+\alpha)} j^{\alpha-1} e^{-\frac{\lambda}{n}} = C_j^{\frac{\lambda}{n}} \quad \text{as } j \rightarrow \infty,$$

where $C_j^{\frac{\lambda}{n}}$ is from (52), see [23, p. 24]. Hence for any $\epsilon > 0$ there exists some positive integer N such that

$$(62) \quad (1 - \epsilon) C_j^{\frac{\lambda}{n}} < \omega_j^{\frac{\lambda}{n}} < (1 + \epsilon) C_j^{\frac{\lambda}{n}}$$

for all $j > N$. It follows that

$$\begin{aligned} (63) \quad \sum_{j=0}^{\infty} |\omega_j^{\frac{\lambda}{n}}|^2 &\leq \left[\sum_{j=0}^N |\omega_j^{\frac{\lambda}{n}}|^2 + (1 + \epsilon)^2 \sum_{j=N+1}^{\infty} |C_j^{\frac{\lambda}{n}}|^2 \right] \\ &\leq \left[\sum_{j=0}^N |\omega_j^{\frac{\lambda}{n}}|^2 + (1 + \epsilon)^2 \sum_{j=0}^{\infty} |C_j^{\frac{\lambda}{n}}|^2 \right] \end{aligned}$$

and consequently,

$$\begin{aligned}
(64) \quad \mathbb{E} \left[\exp \{ i \theta \Delta_1^{-\alpha, \frac{\lambda}{n}} Z_t \} \right] &= \exp \left\{ -\theta^2 \sigma^2 \sum_{j=0}^{\infty} \left| \omega_j^{\frac{\lambda}{n}} \right|^2 \right\} \\
&\geq C_1 \exp \left\{ -(1 + \epsilon^2) \theta^2 \sigma^2 \sum_{j=0}^{\infty} \left| C_j^{\frac{\lambda}{n}} \right|^2 \right\} \\
&= C_1 \mathbb{E} \left[\exp \{ i(1 + \epsilon) \theta X_t^{\frac{\lambda}{n}} \} \right],
\end{aligned}$$

where $C_1 = \exp \left\{ -\theta^2 \sigma^2 \sum_{j=0}^N \left| \omega_j^{\frac{\lambda}{n}} \right|^2 \right\}$ is a finite positive constant. Similarly,

$$(65) \quad \sum_{j=0}^{\infty} \left| \omega_j^{\frac{\lambda}{n}} \right|^2 \geq \left[\sum_{j=0}^N \left| \omega_j^{\frac{\lambda}{n}} \right|^2 + (1 - \epsilon)^2 \sum_{j=N+1}^{\infty} \left| C_j^{\frac{\lambda}{n}} \right|^2 \right],$$

so that

$$\begin{aligned}
(66) \quad \mathbb{E} \left[\exp \{ i \theta \Delta_1^{-\alpha, \frac{\lambda}{n}} Z_t \} \right] &= \exp \left\{ -\theta^2 \sigma^2 \sum_{j=0}^{\infty} \left| \omega_j^{\frac{\lambda}{n}} \right|^2 \right\} \\
&\leq C_2 \exp \left\{ -(1 - \epsilon^2) \theta^2 \sigma^2 \sum_{j=0}^{\infty} \left| C_j^{\frac{\lambda}{n}} \right|^2 \right\} \\
&= C_2 \mathbb{E} \left[\exp \{ i(1 - \epsilon) \theta X_t^{\frac{\lambda}{n}} \} \right],
\end{aligned}$$

where $C_2 = \exp \left\{ -\theta^2 \sigma^2 \sum_{j=0}^N \left| \omega_j^{\frac{\lambda}{n}} \right|^2 \right\}$ is a finite positive constant. From (64) and (66) we have :

$$(67) \quad C_1 \mathbb{E} \left[\exp \{ i(1 + \epsilon) \theta X_t^{\frac{\lambda}{n}} \} \right] \leq \mathbb{E} \left[\exp \{ i \theta \Delta_1^{-\alpha, \frac{\lambda}{n}} Z_t \} \right] \leq C_2 \mathbb{E} \left[\exp \{ i(1 - \epsilon) \theta X_t^{\frac{\lambda}{n}} \} \right]$$

for any $\epsilon > 0$. The proof now follows from (67) and Theorem 5.2 and Theorem (5.3) by letting $\epsilon \rightarrow 0$. \square

Next, we answer the second question that we had in the introduction. Our approach follows that of Pipiras and Taqqu [28].

For $m \in \mathbb{N} \cup \{\infty\}$, we define the approximation

$$\begin{aligned}
f_{n,m}^+ &= \sum_{j=0}^m f\left(\frac{j}{n}\right) 1_{[\frac{j}{n}, \frac{(j+1)}{n}]}, & f_{n,m}^- &= \sum_{j=-m}^{-1} f\left(\frac{j}{n}\right) 1_{[\frac{j}{n}, \frac{(j+1)}{n}]}, \\
f_n^+ &= f_{n,\infty}^+, & f_n^- &= f_{n,\infty}^-, & f_m &= f_n^+ + f_n^-.
\end{aligned}$$

The following theorem answers the third question that we had in the introduction.

Theorem 5.5. *Let $\alpha > 0$ and X_j^λ be the ARTFIMA $(0, \alpha, \lambda, 0)$ times series. Suppose also that the following, condition A, is satisfied:*

Condition A: $f, f_n^\pm \in \mathcal{A}_2, \|f_n^\pm - f_{n,m}^\pm\|_{\mathcal{A}_2} \rightarrow 0$ as $m \rightarrow \infty, \|f - f_n\|_{\mathcal{A}_2} \rightarrow 0$ as $n \rightarrow \infty$.

Then,

$$n^{-H} \sum_{k=0}^{+\infty} f\left(\frac{k}{n}\right) X_k^{\frac{\lambda}{n}} \rightarrow \int_{\mathbb{R}} f(u) Z_{H,\lambda}^1(du)$$

in distribution as $n \rightarrow \infty$.

Proof. For the proof, we suppose

$$W_n^{\frac{\lambda}{n}} = \frac{1}{n^{\frac{1}{2}+\alpha}} \sum_{j=-\infty}^{+\infty} f\left(\frac{j}{n}\right) X_j^{\frac{\lambda}{n}}, \quad W = \int_{\mathbb{R}} f(u) Z_{H,\lambda}^1(du).$$

The Wiener integral W is well-defined, since $f \in \mathcal{A}_2$. To show that the series $W_n^{\frac{\lambda}{n}}$ is well-defined, apply the spectral representation of $X_k^{\frac{\lambda}{n}}$ by Lemma 4.5 and write

$$\begin{aligned} \frac{1}{n^{\frac{1}{2}+\alpha}} \sum_{j=0}^m f\left(\frac{j}{n}\right) X_j^{\frac{\lambda}{n}} &= \frac{1}{n^{\frac{1}{2}+\alpha}} \int_{-\pi}^{\pi} \left(\sum_{j=0}^m f\left(\frac{j}{n}\right) e^{ijx} \right) dZ_{\frac{\lambda}{n}}(x) \\ &= \frac{1}{n^{\frac{1}{2}+\alpha}} \int_{\mathbb{R}} \left(\sum_{j=0}^m f\left(\frac{j}{n}\right) e^{\frac{ij\omega}{n}} \right) 1_{[-\pi n, \pi n]}(\omega) dZ_{\frac{\lambda}{n}}\left(\frac{\omega}{n}\right) \\ &= \frac{1}{n^\alpha - \frac{1}{2}} \int_{\mathbb{R}} \left(\sum_{j=0}^m f\left(\frac{j}{n}\right) \frac{e^{\frac{i(j+1)\omega}{n}} - e^{\frac{ij\omega}{n}}}{i\omega} \right) \frac{\frac{i\omega}{n}}{e^{\frac{i\omega}{n}} - 1} 1_{[-\pi n, \pi n]}(\omega) dZ_{\frac{\lambda}{n}}\left(\frac{\omega}{n}\right) \\ &= \frac{1}{n^\alpha - \frac{1}{2}} \int_{\mathbb{R}} \widehat{f_{n,m}}(\omega) \frac{\frac{i\omega}{n}}{e^{\frac{i\omega}{n}} - 1} 1_{[-\pi n, \pi n]}(\omega) dZ_{\frac{\lambda}{n}}\left(\frac{\omega}{n}\right) \end{aligned}$$

and hence we get

$$\begin{aligned} \mathbb{E} \left| \frac{1}{n^{\frac{1}{2}+\alpha}} \sum_{j=0}^m f\left(\frac{j}{n}\right) X_j^{\frac{\lambda}{n}} \right|^2 &= \int_{\mathbb{R}} \left| \widehat{f_{n,m}}(\omega) \right|^2 \left| \frac{\frac{i\omega}{n}}{e^{\frac{i\omega}{n}} - 1} \right|^2 1_{[-\pi n, \pi n]}(\omega) \frac{1}{n^{2\alpha}} h_{\frac{\lambda}{n}}\left(\frac{\omega}{n}\right) d\omega \\ &\leq C \int_{\mathbb{R}} \left| \widehat{f_{n,m}}(\omega) \right|^2 \frac{1}{n^{2\alpha}} \left(\frac{\lambda^2}{n^2} + \frac{\omega^2}{n^2} \right)^{-\alpha} d\omega. \\ &= C \|f_{n,m}^+\|_{\mathcal{A}_2}^2 \end{aligned}$$

Then, by Condition A,

$$\mathbb{E} \left| \frac{1}{n^{\frac{1}{2}+\alpha}} \sum_{j=m_1+1}^{m_2} f\left(\frac{j}{n}\right) X_j^{\frac{\lambda}{n}} \right|^2 \leq C \|f_{n,m_2}^+ - f_{n,m_1}^+\|_{\mathcal{A}_2}^2 \rightarrow 0$$

as $m_1, m_2 \rightarrow \infty$.

We next show that $W_n^{\frac{\lambda}{n}}$ convergence to W in distribution. Recall from Theorem 3.6 that the set of elementary functions are dense in \mathcal{A}_2 and then there exists a sequence of elementary functions f^l such that $\|f^l - f\|_{\mathcal{A}_2} \rightarrow 0$, as $l \rightarrow \infty$. Now, assume

$$W_n^{\frac{\lambda}{n},l} = \frac{1}{n^{\frac{1}{2}+\alpha}} \sum_{j=-\infty}^{+\infty} f^l\left(\frac{j}{n}\right) X_j^{\frac{\lambda}{n}}, \quad W^l = \int_{\mathbb{R}} f^l(u) Z_{H,\lambda}^1(du).$$

Observe that $W_n^{\frac{\lambda}{n},l}$ is well-defined, since $W_n^{\frac{\lambda}{n},l}$ has a finite number elements and the elementary function f^l is in \mathcal{A}_2 . According to Theorem 4.2. in Billingsley [3], the series $W_n^{\frac{\lambda}{n}}$ convergence in distribution to W if

Step 1: $W^l \rightarrow W$, as $l \rightarrow \infty$,

Step 2: for all $l \in \mathbb{N}$, $W_n^{\frac{\lambda}{n},l} \rightarrow W^l$, as $n \rightarrow \infty$,

Step 3: $\limsup_l \limsup_n \mathbb{E} \left| W_n^{\frac{\lambda}{n},l} - W_n^{\frac{\lambda}{n}} \right|^2 = 0$.

Step 1: The random variables W^l and W have normal distribution with mean zero and finite variance $\|f^l\|_{\mathcal{A}_2}$ and $\|f\|_{\mathcal{A}_2}$, respectively (See Theorem 3.6 and Definition 3.7). Therefore $\mathbb{E} \left| W^l - W \right|^2 = \|f^l - f\|_{\mathcal{A}_2}^2 \rightarrow 0$ as $l \rightarrow \infty$.

Step 2: Observe that $W_n^{\frac{\lambda}{n},l} = \int_{\mathbb{R}} f^l(u) T_n^{\frac{\lambda}{n}}(du)$. Because f^l is an elementary function, then the integral $W_n^{\frac{\lambda}{n},l}$ depends on the process $T_n^{\frac{\lambda}{n}}$ through a finite number of the points only. Now, Theorem 5.2 and Theorem 5.4 imply that $W_n^{\frac{\lambda}{n},l} \rightarrow W^l$, in distribution, as $n \rightarrow \infty$, for all $l \in \mathbb{N}$.

Step 3: For this step, we follow the same way as Pipiras and Taqqu did in Theorem 3.2 [28]. We have $\mathbb{E} \left| W_n^{\frac{\lambda}{n},l} - W_n^{\frac{\lambda}{n}} \right|^2 \leq C \|f_n^l - f_n\|_{\mathcal{A}_2}^2$, where

$$f_n^l := \sum_j f^l\left(\frac{j}{n}\right) 1_{\left(\frac{j}{n}, \frac{(j+1)}{n}\right)}(u).$$

Note that f^l is an elementary function and therefore \widehat{f}_n^l converges to \widehat{f}^l at every point and $\left| \widehat{f}_n^l(\omega) - \widehat{f}^l(\omega) \right| \leq \widehat{g}^l(\omega)$ uniformly in n , for some function $\widehat{g}^l(\omega)$ which is bounded by C_1 and $C_2|\omega|^{-1}$ for all $\omega \in \mathbb{R}$ (See Theorem 3.2. in [28] for more details). Let $\mu^\alpha(d\omega) = |\omega|^{-2\alpha}d\omega$ and $\mu_\lambda^\alpha(d\omega) = |\lambda^2 + \omega^2|^{-2\alpha}d\omega$ be the measures on the real line for $\alpha > 0$. Then apply the dominated converges theorem to see that

$$\begin{aligned} \|f_n^l - f^l\|_{\mathcal{A}_2}^2 &= \|\widehat{f}_n^l - \widehat{f}^l\|_{L^2(\mathbb{R}, \mu_\lambda^\alpha)}^2 \\ &\leq \|\widehat{f}_n^l - \widehat{f}^l\|_{L^2(\mathbb{R}, \mu^\alpha)}^2 \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$. Hence by Condition A, the $\limsup_n \mathbb{E} \left| W_n^{\frac{\lambda}{n},l} - W_n^{\frac{\lambda}{n}} \right|^2 \leq C \|f^l - f\|_{\mathcal{A}_2}^2 \rightarrow 0$ as $l \rightarrow \infty$ and this completes the proof. \square

Remark 5.6. The result of Theorem 5.5 can also be derived by the following condition:

Condition B : $f, f_n^\pm \in \mathcal{A}_1, \|f_n^\pm - f_{n,m}^\pm\|_{\mathcal{A}_1} \rightarrow 0$ as $m \rightarrow \infty, \|f - f_n\|_{\mathcal{A}_1} \rightarrow 0$ as $n \rightarrow \infty$.

Since we have

$$\begin{aligned} \langle f, g \rangle_{\mathcal{A}_1} &= \Gamma(H - \frac{1}{2})^2 \langle \varphi_f, \varphi_g \rangle_2 = \Gamma(1 - \alpha)^2 \langle \hat{\varphi}_f, \hat{\varphi}_g \rangle_2 \\ &= \Gamma(1 - \alpha)^2 \int_{-\infty}^{+\infty} \hat{f}(\omega) \overline{\hat{g}(\omega)} (\lambda^2 + \omega^2)^{\frac{1}{2} - H} d\omega = \langle f, g \rangle_{\mathcal{A}_2} \end{aligned}$$

by the Plancherel Theorem.

6. APPENDIX

Here we recall the definitions of tempered fractional integrals and derivatives and their properties that we used in the pervious sections.

Definition 6.1. For any $f \in L^p(\mathbb{R})$ (where $1 \leq p < \infty$), the positive and negative tempered fractional integrals are defined by

$$(68) \quad \mathbb{I}_+^{\alpha, \lambda} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{+\infty} f(u) (t - u)_+^{\alpha-1} e^{-\lambda(t-u)_+} du$$

and

$$(69) \quad \mathbb{I}_-^{\alpha, \lambda} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} f(u) (u - t)_+^{\alpha-1} e^{-\lambda(u-t)_+} du$$

respectively, for any $\alpha > 0$ and $\lambda > 0$, where $\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} dx$ is the Euler gamma function, and $(x)_+ = xI(x > 0)$.

When $\lambda = 0$ these definitions reduce to the (positive and negative) Riemann-Liouville fractional integral [23, 26, 32], which extends the usual operation of iterated integration to a fractional order. When $\lambda = 1$, the operator (68) is called the Bessel fractional integral [32, Section 18.4].

We state the following lemma without the proof. We refer the reader to see Lemma 2.2 in [22].

Lemma 6.2. For any $\alpha > 0$, $\lambda > 0$, and $p \geq 1$, $\mathbb{I}_\pm^{\alpha, \lambda}$ is a bounded linear operator on $L^p(\mathbb{R})$ such that

$$(70) \quad \|\mathbb{I}_\pm^{\alpha, \lambda} f\|_p \leq \lambda^{-\alpha} \|f\|_p$$

for all $f \in L^p(\mathbb{R})$.

Next we discuss the relationship between tempered fractional integrals and Fourier transforms. Recall that the Fourier transform

$$\mathcal{F}[f](\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\omega x} f(x) dx$$

for functions $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ can be extended to an isometry (a linear onto map that preserves the inner product) on $L^2(\mathbb{R})$ such that

$$(71) \quad \hat{f}(\omega) = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-n}^n e^{-i\omega x} f(x) dx$$

for any $f \in L^2(\mathbb{R})$, see for example [16, Theorem 6.6.4].

Lemma 6.3. *For any $\alpha > 0$ and $\lambda > 0$ we have*

$$(72) \quad \mathcal{F}[\mathbb{I}_{\pm}^{\alpha, \lambda} f](\omega) = \hat{f}(\omega)(\lambda \pm i\omega)^{-\alpha}$$

for all $f \in L^1(\mathbb{R})$ and all $f \in L^2(\mathbb{R})$.

Proof. See Lemma 6.6 in [22]. □

Next we consider the inverse operator of the tempered fractional integral, which is called a tempered fractional derivative. For our purposes, we only require derivatives of order $0 < \alpha < 1$, and this simplifies the presentation.

Definition 6.4. *The positive and negative tempered fractional derivatives of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ are defined as*

$$(73) \quad \mathbb{D}_+^{\alpha, \lambda} f(t) = \lambda^\alpha f(t) + \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^t \frac{f(t) - f(u)}{(t-u)^{\alpha+1}} e^{-\lambda(t-u)} du.$$

and

$$(74) \quad \mathbb{D}_-^{\alpha, \lambda} f(t) = \lambda^\alpha f(t) + \frac{\alpha}{\Gamma(1-\alpha)} \int_t^{+\infty} \frac{f(t) - f(u)}{(u-t)^{\alpha+1}} e^{-\lambda(u-t)} du$$

respectively, for any $0 < \alpha < 1$ and any $\lambda > 0$.

If $\lambda = 0$, the definitions (73) and (74) reduce to the positive and negative Marchaud fractional derivatives [32, Section 5.4].

Note that tempered fractional derivatives cannot be defined pointwise for all functions $f \in L^p(\mathbb{R})$, since we need $|f(t) - f(u)| \rightarrow 0$ fast enough to counter the singularity of the denominator $(t-u)^{\alpha+1}$ as $u \rightarrow t$. We can extend the definition of tempered fractional derivatives to a suitable class of functions in $L^2(\mathbb{R})$. For any $\alpha > 0$ and $\lambda > 0$ we may define the fractional Sobolev space

$$(75) \quad W^{\alpha, 2}(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (\lambda^2 + \omega^2)^\alpha |\hat{f}(\omega)|^2 d\omega < \infty\},$$

which is a Banach space with norm $\|f\|_{\alpha, \lambda} = \|(\lambda^2 + \omega^2)^{\alpha/2} \hat{f}(\omega)\|_2$. The space $W^{\alpha, 2}(\mathbb{R})$ is the same for any $\lambda > 0$ (typically we take $\lambda = 1$) and all the norms $\|f\|_{\alpha, \lambda}$ are equivalent, since $1 + \omega^2 \leq \lambda^2 + \omega^2 \leq \lambda^2(1 + \omega^2)$ for all $\lambda \geq 1$, and $\lambda^2 + \omega^2 \leq 1 + \omega^2 \leq \lambda^{-2}(1 + \omega^2)$ for all $0 < \lambda < 1$.

Definition 6.5. *The positive (resp., negative) tempered fractional derivative $\mathbb{D}_{\pm}^{\alpha,\lambda} f(t)$ of a function $f \in W^{\alpha,2}(\mathbb{R})$ is defined as the unique element of $L^2(\mathbb{R})$ with Fourier transform $\hat{f}(\omega)(\lambda \pm i\omega)^\alpha$ for any $\alpha > 0$ and any $\lambda > 0$.*

Lemma 6.6. *For any $\alpha > 0$ and $\lambda > 0$, we have*

$$(76) \quad \mathbb{D}_{\pm}^{\alpha,\lambda} \mathbb{I}_{\pm}^{\alpha,\lambda} f(t) = f(t)$$

for any function $f \in L^2(\mathbb{R})$, and

$$(77) \quad \mathbb{I}_{\pm}^{\alpha,\lambda} \mathbb{D}_{\pm}^{\alpha,\lambda} f(t) = f(t)$$

for any $f \in W^{\alpha,2}(\mathbb{R})$.

Proof. Given $f \in L^2(\mathbb{R})$, note that $g(t) = \mathbb{I}_{\pm}^{\alpha,\lambda} f(t)$ satisfies $\hat{g}(k) = \hat{f}(\omega)(\lambda \pm i\omega)^{-\alpha}$ by Lemma 6.3, and then it follows easily that $g \in W^{\alpha,2}(\mathbb{R})$. Definition 6.5 implies that

$$(78) \quad \mathcal{F}[\mathbb{D}_{\pm}^{\alpha,\lambda} \mathbb{I}_{\pm}^{\alpha,\lambda} f](\omega) = \mathcal{F}[\mathbb{D}_{\pm}^{\alpha,\lambda} g](\omega) = \hat{g}(\omega)(\lambda \pm i\omega)^\alpha = \hat{f}(\omega),$$

and then (76) follows using the uniqueness of the Fourier transform. The proof of (77) is similar. \square

Here we collect some well known facts about the modified Bessel function of the second kind and we refer the reader to (Chapter 9, [1]) for more details. The modified Bessel function $K_\nu(x)$ is regular function of x . It satisfies the following simple inequality

$$K_\nu(x) > 0 \quad \text{for all } x > 0, \text{ for all } \nu \in \mathbb{R}$$

and it has the following asymptotic expansion:

$$K_\nu(x) \sim 2^{|\nu|-1} \Gamma(|\nu|) x^{-|\nu|} \quad (\nu \neq 0)$$

as $x \rightarrow 0$.

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